Available online at www.sciencedirect.com
SCIENCE DIRECT®

Linear Algebra and its Applications xxx (2006) xxx–xxx

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Generalized matrix diagonal stability and linear dynamical systems

Octavian Pastravanu*, Mihail Voicu

Department of Automatic Control and Industrial Informatics, Technical University "Gh. Asachi" of Iasi, Blvd. Mangeron 53A, Iasi 700050, Romania

Received 21 December 2005; accepted 26 April 2006

Submitted by H. Schneider

Abstract

Let $A = (a_{ij})$ be a real square matrix and $1 \leqslant p \leqslant \infty$. We present two analogous developments. One for Schur stability and the discrete-time dynamical system $x(t+1) = Ax(t)$, and the other for Hurwitz stability and the continuous-time dynamical system $\dot{x}(t) = Ax(t)$. Here is a description of the latter development.

For A, we define and study "Hurwitz diagonal stability with respect to p-norms", abbreviated as "HDS_p". HDS_2 is the usual concept of diagonal stability. A is HDS_p implies "$\mathrm{Re}\,\lambda < 0$ for every eigenvalue λ of A", which means A is "Hurwitz stable", abbreviated as "HS". When the off-diagonal elements of A are nonnegative, A is HS *iff* A is HDS_p for all p.

For the dynamical system $\dot{x}(t) = Ax(t)$, we define "diagonally invariant exponential stability relative to the p-norm", abbreviated as DIES_p, meaning there exist time-dependent sets, which decrease exponentially and are invariant with respect to the system. We show that DIES_p is a special type of exponential stability and the dynamical system has this property *iff* A is HDS_p.

AMS classification: 15A45; 15A60; 15A48; 15A18; 34A30; 34C14; 34D20; 93C05; 93D20

Keywords: Matrix diagonal stability; Stein/Lyapunov matrix inequality; (essentially) Nonnegative matrices; Gershgorin's disks; Linear dynamical systems; Invariant sets; (diagonally invariant) Exponential stability

* Corresponding author. Tel./fax: +40 232 230751.
E-mail addresses: opastrav@ac.tuiasi.ro (O. Pastravanu), mvoicu@ac.tuiasi.ro (M. Voicu).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2006.04.021

1. Introduction

First, let us introduce the following notations:

- For a vector $x \in \mathbf{R}^n$:
  $\|x\|$ is an arbitrary vector norm;
  $\|x\|_p$ is the Hölder vector p-norm and $1 \leqslant p \leqslant \infty$;
  $\|x\|_p^D = \|D^{-1}x\|_p$, where D is a positive definite diagonal matrix;
  $|x|$ denotes a nonnegative vector defined by taking the absolute values of the elements of x.
- For a matrix $M \in \mathbf{R}^{n\times n}$:
  $\|M\|$ is the matrix norm induced by the vector norm $\|\bullet\|$;
  $\|M\|_p$ is the matrix norm induced by the vector norm $\|\bullet\|_p$;
  $\|M\|_p^D = \|D^{-1}MD\|_p$ is the matrix norm induced by the vector norm $\|\bullet\|_p^D$;
  $m_p^D(M) = \lim_{h\downarrow 0}(\|I + hM\|_p^D - 1)/h$ is a matrix measure [1, p. 41], based on the matrix norm $\|\bullet\|_p^D$;
  $\sigma(M) = \{z \in \mathbf{C}|\det(zI - M) = 0\}$ is the spectrum of M, and $\lambda_i(M) \in \sigma(M)$, $i = 1, \ldots, n$, denotes its eigenvalues;
  $|M|$ denotes a nonnegative matrix defined by taking the absolute values of the entries of M.

If $x, y \in \mathbf{R}^n$, then "$x \leqslant y$" and "$x < y$" mean componentwise inequalities.

If $M, P \in \mathbf{R}^{n\times n}$, then "$M \leqslant P$", "$M < P$" mean componentwise inequalities.

We shall write "X // Y" in place of "X [respectively Y]".

In the complex plane $\mathbf{C}$, define the regions $\mathbf{C}_S = \{z \in \mathbf{C}||z| < 1\}$ // $\mathbf{C}_H = \{z \in \mathbf{C}|\mathrm{Re}\, z < 0\}$. If $\sigma(M) \subset \mathbf{C}_S$ // $\mathbf{C}_H$, then $M \in \mathbf{R}^{n\times n}$ is said to be *Schur stable* (abbreviated as SS) // *Hurwitz stable* (abbreviated as HS).

If $M \in \mathbf{R}^{n\times n}$ is symmetric, then "$M \succ 0$" // "$M \prec 0$" means M is positive definite // negative definite.

Throughout the text, $A = (a_{ij})$ denotes a real n by n matrix.

"Matrix diagonal stability" is defined in [2] as follows: A is Schur // Hurwitz diagonally stable if there exists a diagonal matrix $P \succ 0$, such that

$$A^{\mathrm{T}}PA - P \prec 0 \text{ // } A^{\mathrm{T}}P + PA \prec 0. \tag{1-S//H}$$

For these concepts, we propose the following generalizations:

Definition 1. A is called *Schur // Hurwitz diagonally stable relative to the p-norm* (abbreviated SDS_p // HDS_p) if there exists a diagonal matrix $D \succ 0$, such that

$$\|A\|_p^D < 1 \text{ // } m_p^D(A) < 0. \tag{2-S//H}$$

In the remainder of the text we shall also use the abbreviation SDS_p // HDS_p to mean "Schur // Hurwitz diagonal stability relative to the p-norm".

Remark 1. Set $P = (D^{-1})^2$. When $p = 2$, inequality (2-S//H) is equivalent to the Stein // Lyapunov matrix inequality (1-S//H).

Remark 2. If $1 \leqslant p \leqslant \infty$ and A is SDS_p // HDS_p, then A is SS // HS. Indeed, $\sigma(A) = \sigma(D^{-1}AD)$ and we can denote the eigenvalues of A and $(D^{-1}AD)$, such that $\lambda_i(A) = \lambda_i(D^{-1}AD)$, $i = 1, \ldots, n$. Thus $|\lambda_i(A)| = |\lambda_i(D^{-1}AD)| \leqslant \|A\|_p^D < 1$ and $\mathrm{Re}\{\lambda_i(A)\} = \mathrm{Re}\{\lambda_i(D^{-1}AD)\} \leqslant m_p^D(A) < 0$, $i = 1, \ldots, n$ [1, p. 41].

We first analyze SDS_p // HDS_p as a matrix property. Then we explore the connections between SDS_p // HDS_p and the behavior of a linear dynamical system with *discrete-time // continuous-time* (abbreviated as DT // CT) dynamics, defined by

$$x(t+1) = Ax(t) \;//\; \dot{x}(t) = Ax(t)$$
$$\text{for } t, t_0 \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\} \;//\; \mathbf{R}_+ = \{\tau \in \mathbf{R} | \tau \geqslant 0\}, \text{ and } t \geqslant t_0,$$
$$\text{with the initial condition } x(t_0) = x_0 \in \mathbf{R}^n. \tag{3-S//H}$$

The remainder of the paper is organized as follows:

Section 2 provides some useful results about nonnegative // essentially nonnegative matrices.

Section 3 presents SDS_p // HDS_p criteria that rely on

- a test matrix A^S // A^H built from A as

$$A^S = |A| \;//\; A^H = (a_{ij}^H), \quad \text{where } a_{ij}^H = a_{ii} \text{ if } i = j \text{ and } |a_{ij}| \text{ otherwise;} \tag{4-S//H}$$

- the generalized Gershgorin's disks of A, defined with $D = \mathrm{diag}\{d_1, \ldots, d_n\} \succ 0$, for columns by

$$G_j^c(D^{-1}AD) = \left\{ z \in \mathbf{C} \left| |z - a_{jj}| \leqslant \sum_{i=1, i \neq j}^{n} \frac{d_j}{d_i} |a_{ij}| \right. \right\}, \quad j = 1, \ldots, n, \tag{5}$$

or for rows by

$$G_i^r(D^{-1}AD) = \left\{ z \in \mathbf{C} \left| |z - a_{ii}| \leqslant \sum_{j=1, j \neq i}^{n} \frac{d_j}{d_i} |a_{ij}| \right. \right\}, \quad i = 1, \ldots, n. \tag{6}$$

Section 4 introduces a property called *diagonally invariant exponential stability relative to the p-norm* (abbreviated as DIES_p) of a linear dynamical system (3-S//H). DIES_p ensures the existence of time-dependent sets with DT // CT exponential decrease, which are invariant with respect to (abbreviated w.r.t.) the state-space trajectories (solutions) of system (3-S//H). This means once the initial condition $x(t_0) = x_0$ belongs to such a set, the corresponding solution $x(t; t_0, x_0)$ also belongs to the set, for any $t \geqslant t_0$ (see [3, p. 100]). We show that DIES_p is a special type of *exponential stability* (abbreviated as ES) (see [3, pp. 107–108]) of the dynamical system (3-S//H), fully characterized by A's being in SDS_p // HDS_p.

Section 5 illustrates the applicability of the main results by an example.

2. Preliminary results

2.1. Nonnegative // essentially nonnegative matrices

A real square matrix is called *nonnegative // essentially nonnegative* if its entries // off-diagonal entries are nonnegative. In the following lemmas, we use the notation "S // H" for presenting results that refer to nonnegative // essentially nonnegative matrices, since these results support the approach to SDS_p // HDS_p to be developed in Section 3.

Lemma 1. (S) *If A is nonnegative, its spectral radius $\lambda_{\max}(A)$ is an eigenvalue such that $|\lambda_i(A)| \leqslant \lambda_{\max}(A)$, $i = 1, \ldots, n$.*

(H) *If A is essentially nonnegative, then it has a real eigenvalue, denoted by* $\lambda_{\max}(A)$, *such that* $\mathrm{Re}\{\lambda_i(A)\} \leqslant \lambda_{\max}(A), i = 1, \ldots, n$.

Proof. (S) From Theorem 8.2.2 in [4].

(H) Pick a real s such that $sI + A$ is nonnegative. Denote the eigenvalues of A and $sI + A$ such that $\lambda_i(sI + A) = s + \lambda_i(A)$, $i = 1, \ldots, n$. Define $\lambda_{\max}(A)$ so that $s + \lambda_{\max}(A) = \lambda_{\max}(sI + A) \geqslant |\lambda_i(sI + A)| \geqslant s + \mathrm{Re}\{\lambda_i(A)\}, i = 1, \ldots, n$. □

So whenever both definitions of $\lambda_{\max}(A)$ make sense, they agree.

Lemma 2. *Let* $1 \leqslant p \leqslant \infty$ *and* $D \succ 0$ *diagonal. If* A, B *are nonnegative // essentially nonnegative and* $A \leqslant B$, *then* (S) $\|A\|_p^D \leqslant \|B\|_p^D$ // (H) $m_p^D(A) \leqslant m_p^D(B)$.

Proof. (S) If A, B are nonnegative, then, for any $y \in \mathbf{R}^n$, we can write the componentwise inequalities $|(D^{-1}AD)y| \leqslant |(D^{-1}AD)|y|| \leqslant |(D^{-1}BD)|y||$. From Theorem 5.5.10 in [4], the monotonicity of the p-vector-norms implies $\|(D^{-1}AD)y\|_p \leqslant \|(D^{-1}AD)|y|\|_p \leqslant \|(D^{-1}BD)|y|\|_p$, yielding $\|(D^{-1}AD)y\|_p \leqslant \|D^{-1}BD\|_p \||y|\|_p = \|D^{-1}BD\|_p \|y\|_p$. Consequently, $\|D^{-1}AD\|_p = \max_{\|y\|_p=1} \|(D^{-1}AD)y\|_p \leqslant \|D^{-1}BD\|_p$, which completes the proof of part (S).

(H) If A, B are essentially nonnegative, then, for small $h > 0$, we have the componentwise inequality $0 \leqslant (I + hD^{-1}AD) \leqslant (I + hD^{-1}BD)$ that implies $\|I + hD^{-1}AD\|_p \leqslant \|I + hD^{-1}BD\|_p$, according to part (S). Thus, $(\|D^{-1}(I + hA)D\|_p - 1)/h \leqslant (\|D^{-1}(I + hB)D\|_p - 1)/h$ and taking $h \downarrow 0$, we get $m_p^D(A) \leqslant m_p^D(B)$. □

Lemma 3. *Let* $1 \leqslant p \leqslant \infty$ *and* $r > \lambda_{\max}(A)$, *where A is nonnegative // essentially nonnegative. Then there exists a diagonal matrix* $D \succ 0$ *such that* (S) $\lambda_{\max}(A) \leqslant \|A\|_p^D < r$ // (H) $\lambda_{\max}(A) \leqslant m_p^D(A) < r$.

Proof. (S) Suppose A is nonnegative. If J is the n by n matrix with all its entries 1, then $\lambda_{\max}(A + \varepsilon J)$ as a function of $\varepsilon \geqslant 0$ is continuous and increasing, according to Theorem 8.1.18 in [4]. Hence, for any $r > \lambda_{\max}(A)$, we can find an $\varepsilon^* > 0$ such that $\lambda_{\max}(A + \varepsilon^* J) < r$. On the other hand, the matrix $A + \varepsilon^* J$ is positive and there exists its right and left Perron eigenvectors $v = [v_1 \ldots v_n]^{\mathrm{T}} > 0$ and $w = [w_1 \cdots w_n]^{\mathrm{T}} > 0$, respectively. If $1/p + 1/q = 1$, then, from [5] we have $\|D^{-1}(A + \varepsilon^* J)D\|_p = \lambda_{\max}(A + \varepsilon^* J)$ with $D = \mathrm{diag}\{v_1^{1/q}/w_1^{1/p}, \ldots, v_n^{1/q}/w_n^{1/p}\}$, where the particular cases of norms $p = 1$ and $p = \infty$ mean $1/p = 1$, $1/q = 0$, and $1/p = 0$, $1/q = 1$, respectively. Since $0 \leqslant A < A + \varepsilon^* J$, by using Lemma 2 we can write $\|D^{-1}AD\|_p \leqslant \|D^{-1}(A + \varepsilon^* J)D\|_p$ and, finally, we get $\lambda_{\max}(A) \leqslant \|D^{-1}AD\|_p = \|A\|_p^D < r$.

(H) Suppose A is essentially nonnegative. Consider an arbitrary $s > 0$ such that $sI + A$ is nonnegative. Choose $\tilde{r}$ such that $\lambda_{\max}(A) < \tilde{r} < r$. By using part (H) with $\tilde{r}$ instead of r and taking into account that the eigenvectors of $sI + A + \varepsilon^* J$ and $A + \varepsilon^* J$ are identical, we find a matrix $D \succ 0$ diagonal, such that $s + \lambda_{\max}(A) = \lambda_{\max}(sI + A) \leqslant \|sI + A\|_p^D < s + \tilde{r}$. For $s = 1/h$, we have $\lambda_{\max}(A) \leqslant (\|I + hA\|_p^D - 1)/h < \tilde{r}$ that, when $h \downarrow 0$, yields $\lambda_{\max}(A) \leqslant m_p^D(A) \leqslant \tilde{r} < r$. □

Remark 3. Let $1 \leqslant p \leqslant \infty$. For A irreducible (i.e. the oriented graph associated with A is strongly connected; other equivalent characterizations are given by Theorem 6.2.24 in [4]), we have a particular case of Lemma 3. If A is nonnegative // essentially nonnegative, then there exists a diagonal matrix $D \succ 0$ such that $\lambda_{\max}(A) = \|A\|_p^D$ // $\lambda_{\max}(A) = m_p^D(A)$. The matrix D is built by the procedure presented in the proof of Lemma 3, applied to the right and left Perron–Frobenius eigenvectors of A (which are positive, since A is irreducible – see Theorem 8.4.4 in [4]).

2.2. *Matrices majorized by nonnegative // essentially nonnegative matrices*

The matrix A^S // A^H defined by (4-S//H) is nonnegative // essentially nonnegative and majorizes A.

Lemma 4. *If* $1 \leqslant p \leqslant \infty$ *and* $D \succ 0$ *is diagonal, then* (S) $\|A\|_p^D \leqslant \|A^S\|_p^D$ // (H) $m_p^D(A) \leqslant m_p^D(A^H)$.

Proof. (S) For any $y \in \mathbf{R}^n$, we can write the componentwise inequality $|(D^{-1}AD)y| \leqslant |(D^{-1}A^S D)|y\|$, and the monotonicity of the p-vector-norms (Theorem 5.5.10 in [4]) yields $\|(D^{-1}AD)y||_p \leqslant \|(D^{-1}A^S D)|y|\|_p \leqslant \|D^{-1}A^S D\|_p\||y|\|_p = \|D^{-1}A^S D\|_p\|y\|_p$. Hence, $\|D^{-1}AD\|_p = \max_{\|y\|_p=1} \|(D^{-1}AD)y\|_p \leqslant \|D^{-1}A^S D\|_p$, which completes the proof of part (S).

(H) For small $h > 0$, we have $|I + hD^{-1}AD| = (I + hD^{-1}A^H D)$ and we can apply part (S), obtaining $\|I + hA\|_p^D = \|I + hD^{-1}AD\|_p \leqslant \|I + hD^{-1}A^H D\|_p = \|I + hA^H\|_p^D \Rightarrow (\|I + hA\|_p^D - 1)/h \leqslant (\|I + hA^H\|_p^D)/h$. Taking $h \downarrow 0$, we get $m_p^D(A) \leqslant m_p^D(A^H)$. □

Remark 4. If $p = 1, \infty$ and $D \succ 0$ is diagonal, we have a particular case of Lemma 4, namely, (S) $\|A\|_p^D = \|A^S\|_p^D$, (H) $m_p^D(A) = m_p^D(A^H)$. Indeed, for any matrix $M \in \mathbf{R}^{n\times n}$, $\|M\|_p = \||M|\|_p$ if $p = 1, \infty$, and taking into account the equality (S) $|D^{-1}AD| = D^{-1}A^S D$, (H) $|I + hD^{-1}AD| = I + hD^{-1}A^H D$ for small $h > 0$, we get (S) $\|D^{-1}AD\|_p = \|D^{-1}A^S D\|_p$, (H) $(\|I + hD^{-1}AD\|_p - 1)/h = (\|I + hD^{-1}A^H D\|_p - 1)/h$, respectively.

3. SDS$_p$ // HDS$_p$ criteria

Theorem 1. *The following six statements are equivalent*:

(i) A^S // A^H *is* SS // HS.
(ii) A *is* SDS$_1$ // HDS$_1$.
(iii) A *is* SDS$_\infty$ // HDS$_\infty$.
(iv) *There exists a diagonal matrix* $D \succ 0$, *such that* $\bigcup_{j=1}^n G_j^c(D^{-1}AD) \subseteq \mathbf{C}_S$ // $\mathbf{C}_H$.
(v) *There exists a diagonal matrix* $D \succ 0$, *such that* $\bigcup_{i=1}^n G_i^r(D^{-1}AD) \subseteq \mathbf{C}_S$ // $\mathbf{C}_H$.
(vi) A *is* SDS$_p$ // HDS$_p$ *for all* $1 \leqslant p \leqslant \infty$.

Proof. (S) (i) ⇒ (vi). Let $1 \leqslant p \leqslant \infty$. If A^S is SS, then $\lambda_{\max}(A^S) < 1$ and, according to Lemma 3(S), one can find a diagonal matrix $D \succ 0$ such that $\lambda_{\max}(A^S) \leqslant \|A^S\|_p^D < 1$. Then apply Lemma 4(S).

(S) (ii) ⇒ (i) and (iii) ⇒ (i). If A is SDS$_p$, with $p = 1$ or ∞, then by (2-S) there exists a diagonal matrix $D \succ 0$ such that $\|A\|_p^D < 1$ and the spectral radius of A^S is $\lambda_{\max}(A^S) \leqslant \|A^S\|_p^D = \|A\|_p^D$ by Remark 4(S).

(S) (ii) ⇔ (iv). It results from $\|A\|_1^D < 1 \Leftrightarrow -1 < a_{jj} - \sum_{i=1,i\neq j}^n \frac{d_j}{d_i}|a_{ij}|$ and $a_{jj} + \sum_{i=1,i\neq j}^n \frac{d_j}{d_i}|a_{ij}| < 1,\ j = 1, \ldots, n \Leftrightarrow G_j^c(D^{-1}AD) \subseteq \mathbf{C}_S,\ j = 1, \ldots, n$.

(S) (iii) ⇔ (v). It is similar to the proof of (S) (ii) ⇔ (iv).

(H) (i) ⇒ (vi). Let $1 \leqslant p \leqslant \infty$. If A^H is HS, then $\lambda_{\max}(A^H) < 0$ and, according to Lemma 3(H), one can find a diagonal matrix $D \succ 0$ such that $\lambda_{\max}(A^H) \leqslant m_p^D(A^H) < 0$. Then apply Lemma 4(H).

(H) (ii) ⇒ (i) and (iii) ⇒ (i). If A is HDS_p, with $p = 1$ or ∞, then by (2-H) there exists a diagonal matrix $D \succ 0$ such that $m_p^D(A) < 0$ and $\lambda_{\max}(A^H) \leqslant m_p^D(A^H) = m_p^D(A)$ by Remark 4(H).

(H) (ii) ⇔ (iv). It results from $m_1^D(A) < 0 \Leftrightarrow 0 > m_1^D(A) = a_{jj} + \sum_{i=1, i \neq j}^{n} \frac{d_j}{d_i} |a_{ij}|, \ \ j = 1, \ldots, n$ [1, p. 41] $\Leftrightarrow G_j^c(D^{-1}AD) \subset \mathbf{C}_H, \ \ j = 1, \ldots, n$.

(H) (iii) ⇔ (v). It is similar to the proof of (H) (ii) ⇔ (iv).

(S // H) (i) ⇒ (ii) and (i) ⇒ (iii). These are particular cases of the implication (S // H) (i) ⇒ (vi), with $p = 1$ and $p = \infty$, respectively.

(S // H) (vi) ⇒ (ii) and (vi) ⇒ (iii). These are obvious. □

Corollary 1. *If A is nonnegative // essentially nonnegative, then the following five statements are equivalent*:

(i) *A is* SS // HS.
(ii) *There exists a* p, $1 \leqslant p \leqslant \infty$, *such that A is* SDS_p // HDS_p.
(iii) *There exists a diagonal matrix* $D \succ 0$, *such that* $\bigcup_{j=1}^{n} G_j^c(D^{-1}AD) \subseteq \mathbf{C}_S$ // $\mathbf{C}_H$.
(iv) *There exists a diagonal matrix* $D \succ 0$, *such that* $\bigcup_{i=1}^{n} G_i^r(D^{-1}AD) \subseteq \mathbf{C}_S$ // $\mathbf{C}_H$.
(v) *A is* SDS_p // HDS_p *for all* $1 \leqslant p \leqslant \infty$.

Proof. (S) (ii) ⇒ (i). If A is SDS_p, then the spectral radius $\lambda_{\max}(A) \leqslant \|A\|_p^D < 1$.

(H) (ii) ⇒ (i). If A is HDS_p, then $\lambda_{\max}(A) \leqslant m_p^D(A) < 0$.

(S // H) (i) ⇔ (iii) ⇔ (iv) and (i) ⇒ (v). These result from Theorem 1, since $A = A^S$ // A^H.

(S // H) (v) ⇒ (ii). It is obvious. □

Remark 5. Part of the results presented by Theorem 1 and Corollary 1 can also be found in [2]. In terms of our notations, these results are as follows. A is $\mathrm{SDS}_\infty \Leftrightarrow A^S$ is SS (Corollary 2.7.29 in [2]). A^S is SS $\Rightarrow$ A is SDS_2 (Corollary 2.7.27 in [2]). If A is nonnegative, then A is $\mathrm{SDS}_\infty \Leftrightarrow A$ is $\mathrm{SDS}_2 \Leftrightarrow A$ is SS (Lemma 2.7.25 in [2], but its proof does not cover the case of A reducible). If A is essentially nonnegative, then A is $\mathrm{HDS}_2 \Leftrightarrow A$ is HS (Theorem 2.2.1 in [2]). However, [2] remains focused on diagonal stability as considered by that text (i.e. SDS_2 // HDS_2 in our formulation) and does not suggest a generalization of the "diagonal stability" concept to general p-norms.

4. Connections to the dynamics of linear systems

Let $x(t; t_0, x_0)$ denote the solution of (3-S//H) satisfying the initial condition $x(t_0) = x_0$.

Definition 2. The system (3-S//H) is called *diagonally invariant exponentially stable relative to the p-norm* (abbreviated as DIES_p) if there exist a diagonal matrix $D \succ 0$ and a constant $0 < r < 1$ // $r < 0$, such that

$$\forall \varepsilon > 0, \ \forall t, t_0 \in \mathbf{Z}_+ \ // \ \mathbf{R}_+, \ t \geqslant t_0, \ \forall x_0 = x(t_0) \in \mathbf{R}^n,$$
$$\|x_0\|_p^D \leqslant \varepsilon \Rightarrow \|x(t; t_0, x_0)\|_p^D \leqslant \varepsilon r^{(t-t_0)} \ // \ \|x(t; t_0, x_0)\|_p^D \leqslant \varepsilon \mathrm{e}^{r(t-t_0)}. \quad \text{(7-S//H)}$$

In the remainder of the text we shall also use the abbreviation DIES$_p$ to mean "diagonally invariant exponential stability relative to the p-norm".

Remark 6. In terms of invariant sets (see [3, p. 100]), Definition 2 is equivalent to the existence of a diagonal matrix $D \succ 0$ and a constant $0 < r < 1$ // $r < 0$ ensuring that the time-dependent sets

$$X_p^{\varepsilon}(t; t_0) = \left\{ x \in \mathbf{R}^n \| \|x\|_p^D \leqslant \varepsilon r^{(t-t_0)} \text{ // } \|x\|_p^D \leqslant \varepsilon \mathrm{e}^{r(t-t_0)} \right\},$$
$$t, t_0 \in \mathbf{Z}_+ \text{ // } \mathbf{R}_+,\ t \geqslant t_0,\ \varepsilon > 0, \tag{8-S//H}$$

are (positively) invariant w.r.t. the solutions (state-space trajectories) of system (3-S//H). For the usual values $p = 1, 2, \infty$, these sets have well-known geometric shapes (i.e. hyper-diamonds, ellipses and rectangles, respectively), scaled in accordance with the diagonal entries of the matrix D.

Theorem 2. *Let* $1 \leqslant p \leqslant \infty$, $D \succ 0$ *diagonal and* $0 < r < 1$ // $r < 0$. *The following four statements are equivalent*:

(i) *System* (3-S//H) *is* DIES$_p$ *for D and r, i.e.* (7-S//H) *holds.*
(ii) $V(x) = \|x\|_p^D$ *is a strong Lyapunov function for system* (3-S//H), *with the decreasing rate r, i.e.*

$$\forall t \in \mathbf{Z}_+ \text{ // } \mathbf{R}_+,\ \forall \text{ solution } x(t) \text{ to } (3\text{-S//H}),$$
$$V(x(t+1)) \leqslant rV(x(t)) \text{ // } D^+V(x(t)) \leqslant rV(x(t)), \tag{9-S//H}$$

where $D^+V(x(t)) = \lim_{h\downarrow 0}(V(x(t+h)) - V(x(t)))/h$.

(iii) $\forall \tau \in \mathbf{Z}_+ \text{ // } \mathbf{R}_+, \quad \|A^\tau\|_p^D \leqslant r^\tau \text{ // } \|\mathrm{e}^{A\tau}\|_p^D \leqslant \mathrm{e}^{r\tau}.$ (10-S//H)

(iv) $\|A\|_p^D \leqslant r \text{ // } m_p^D(A) \leqslant r.$ (11-S//H)

Proof. (S) (i) ⇒ (ii). Let x solve (3-S) and let $t \in \mathbf{Z}_+$. Set $\varepsilon = \|x(t)\|_p^D = V(x(t))$. If $\varepsilon = 0$, then $x(t)$, and hence $x(t+1)$, is 0. If $\varepsilon > 0$, then by (7-S) $V(x(t+1)) = \|x(t+1; t, x(t))\|_p^D \leqslant V(x(t))r$.

(S) (ii) ⇒ (i). Assume that system (3-S) is not DIES$_p$ for D and r. Let $\tilde{x}(t)$ solve (3-S) and violate the condition (7-S), i.e. there exist $\varepsilon > 0$, $t^*, t_0 \in \mathbf{Z}_+$, $t^* \geqslant t_0$, with $\|\tilde{x}(t^*)\|_p^D \leqslant \varepsilon r^{(t^*-t_0)}$ and $\|\tilde{x}(t^*+1)\|_p^D > \varepsilon r^{(t^*+1-t_0)}$. This means $r\|\tilde{x}(t^*)\|_p^D < \|\tilde{x}(t^*+1)\|_p^D$, which contradicts (9-S).

(S) (ii) ⇒ (iii). $\forall t_0, \tau \in \mathbf{Z}_+$, $\|A^\tau\|_p^D = \sup_{x(t_0)\neq 0} \frac{\|A^\tau x(t_0)\|_p^D}{\|x(t_0)\|_p^D} = \sup_{x(t_0)\neq 0} \frac{\|x(t_0+\tau)\|_p^D}{\|x(t_0)\|_p^D} \leqslant r^\tau$ according to (9-S).

(S) (iii) ⇒ (iv). Set $\tau = 1$ in (10-S).

(S) (iv) ⇒ (ii). By (3-S).

(H) First let us show that (9-H) can be written in the equivalent form

$$\forall t \geqslant t_0 \geqslant 0, \quad \forall x_0 \in \mathbf{R}^n, \quad \|x(t; t_0, x_0)\|_p^D \leqslant \mathrm{e}^{r(t-t_0)}\|x_0\|_p^D. \tag{12}$$

Indeed, if (12) is true, then, for any solution $x(s)$ of (3-H) with initial condition at $s_0 = t$, we have $D^+V(x(s_0)) = \lim_{h\downarrow 0}(\|x(s_0+h; s_0, x_0)\|_p^D - \|x_0\|_p^D)/h \leqslant (\lim_{h\downarrow 0}(\mathrm{e}^{rh}-1)/h)\|x_0\|_p^D = rV(x(s_0))$. Conversely, let $t_0 \geqslant 0$ and $x_0 \in \mathbf{R}^n$. If (9-H) holds for $x(t) = x(t; t_0, x_0)$, consider the differential equation $\dot{y}(t) = ry(t)$ with the initial condition $y(t_0) = V(x(t_0)) = V(x_0)$. Then, for all $t \geqslant t_0$, $V(x(t)) \leqslant y(t) = \mathrm{e}^{r(t-t_0)}y(t_0) = \mathrm{e}^{r(t-t_0)}V(x_0)$, according to Theorem 4.2.11 in [3]. Thus we may use (12) instead of (9-H) in the following proofs.

We also need to show that

$$\lim_{h\downarrow 0}(\|\mathrm{e}^{Ah}\|_p^D - 1)/h = m_p^D(A). \tag{13}$$

Indeed, $\mathrm{e}^{Ah} = I + hA + h\mathrm{O}(h)$, $\lim_{h\downarrow 0}\mathrm{O}(h) = 0$, together with $\|I + hA\|_p^D - h\|\mathrm{O}(h)\|_p^D \leqslant \|I + hA + h\mathrm{O}(h)\|_p^D \leqslant \|I + hA\|_p^D + h\|\mathrm{O}(h)\|_p^D$ yield $(\|I + hA\|_p^D - 1)/h - \|\mathrm{O}(h)\|_p^D \leqslant (\|\mathrm{e}^{Ah}\|_p^D - 1)/h \leqslant (\|I + hA\|_p^D - 1)/h + \|\mathrm{O}(h)\|_p^D$.

(H) (i) ⇒ (ii). By taking $\varepsilon = \|x_0\|_p^D$ in (7-H), we get (12) if $\varepsilon > 0$. If $\varepsilon = 0$, then $x_0 = 0$ and hence $x(t) = \mathrm{e}^{A(t-t_0)}x_0 = 0$, for $t \geqslant t_0$.

(H) (ii) ⇒ (i). Assume that system (3-H) is not DIES$_p$ for D and r. Let $\tilde{x}(t)$ solve (3-H) and violate the condition (7-H), i.e. there exist $\varepsilon > 0$, $t^*, t^{**} \in R_+$, $t^{**} > t^* \geqslant t_0$, with $\|\tilde{x}(t^*)\|_p^D \leqslant \varepsilon\mathrm{e}^{r(t^*-t_0)}$ and $\|\tilde{x}(t^{**})\|_p^D > \varepsilon\mathrm{e}^{r(t^{**}-t_0)}$. This means $\mathrm{e}^{r(t^{**}-t^*)}\|\tilde{x}(t^*)\|_p^D < \|\tilde{x}(t^{**})\|_p^D$, which contradicts (12).

(H) (ii) ⇒ (iii). $\forall t_0, \tau \in \mathbf{R}_+$, $\|\mathrm{e}^{A\tau}\|_p^D = \sup_{x_0\neq 0}\frac{\|\mathrm{e}^{A\tau}x_0\|_p^D}{\|x_0\|_p^D} = \sup_{x_0\neq 0}\frac{\|x(t_0+\tau;t_0,x_0)\|_p^D}{\|x_0\|_p^D} \leqslant \mathrm{e}^{r\tau}$ according to (12).

(H) (iii) ⇒ (iv). It results from (10-H) and (13), since $m_p^D(A) = \lim_{h\downarrow 0}(\|\mathrm{e}^{Ah}\|_p^D - 1)/h \leqslant \lim_{h\downarrow 0}(\mathrm{e}^{rh} - 1)/h = r$.

(H) (iv) ⇒ (ii). $\forall$ solution x to (3-H), $\forall t \in \mathbf{R}_+$, we can write $D^+\|x(t)\|_p^D = \lim_{h\downarrow 0}(\|x(t+h)\|_p^D - \|x(t)\|_p^D)/h = \lim_{h\downarrow 0}(\|\mathrm{e}^{Ah}x(t)\|_p^D - \|x(t)\|_p^D)/h \leqslant [\lim_{h\downarrow 0}(\|\mathrm{e}^{Ah}\|_p^D - 1)/h]\|x(t)\|_p^D = m_p^D(A)\|x(t)\|_p^D \leqslant r\|x(t)\|_p^D$, according to (13). □

We are interested in exploring the relationship between the DIES$_p$ of the linear system (3-S//H) and the classical concept of ES.

Generally speaking, for a DT // CT dynamical system (nonlinear, time-variant), the ES is a property associated with a certain solution (trajectory) $x(t)$. If $x(t) \equiv 0$ is a solution to the considered system, then $\{0\}$ is called an *equilibrium* of the system, and its ES is defined by

Definition 3 (see Definition 3.2.6 in [3]). The equilibrium $\{0\}$ is ES *in the small* if

$$\begin{aligned}&\exists r \in (0,1) \;//\; r < 0 : \forall \alpha > 0, \forall t, t_0 \in \mathbf{Z}_+ \;//\; \mathbf{R}_+, t \geqslant t_0, \exists \delta(\alpha) :\\ &\|x_0\| < \delta(\alpha) \Rightarrow \|x(t;t_0,x_0)\| \leqslant \alpha r^{(t-t_0)} \;//\; \|x(t;t_0,x_0)\| \leqslant \alpha \mathrm{e}^{r(t-t_0)}.\end{aligned} \tag{14-S//H}$$

Definition 4 (see Definition 3.2.13 in [3]). The equilibrium $\{0\}$ is ES *in the large* if

$$\begin{aligned}&\exists r \in (0,1) \;//\; r < 0, \exists \gamma > 0 : \forall \beta > 0, \forall t, t_0 \in \mathbf{Z}_+ \;//\; \mathbf{R}_+, t \geqslant t_0,\ \exists M(\beta) > 0 :\\ &\|x_0\| < \beta \Rightarrow \|x(t;t_0,x_0)\| \leqslant M(\beta)r^{(t-t_0)}\|x_0\|^\gamma \;//\; \|x(t;t_0,x_0)\| \leqslant M(\beta)\mathrm{e}^{r(t-t_0)}\|x_0\|^\gamma.\end{aligned} \tag{15-S//H}$$

In the particular case of the linear time-invariant system (3-S//H), the ES is considered a system property, with a global meaning. Thus, "system (3-S//H) is ES" is equivalent to "the equilibrium $\{0\}$ is ES in the large" (see Examples 3.2.14, 3.2.16 in [3]). The ES of system (3-S//H) is characterized by the following two well known results:

Proposition 1 (see Theorem 1(v) in Chapter III of [1]). *System* (3-S//H) *is ES if and only if the state-transition matrix* (*fundamental matrix*, *or semigroup of linear operators*) A^τ // $\mathrm{e}^{A\tau}$, $\tau \in \mathbf{Z}_+$ // $\mathbf{R}_+$, *fulfils the condition*:

$$\exists r \in (0,1) \;//\; r<0,\ \exists K \geqslant 1 : \forall \tau \in \mathbf{Z}_+ \;//\; \mathbf{R}_+,\ \|A^\tau\| \leqslant K r^\tau \;//\; \|\mathrm{e}^{A\tau}\| \leqslant K\mathrm{e}^{r\tau}. \qquad \text{(16-S//H)}$$

Proposition 2 (see Examples 3.2.14, 3.2.16 in [3]). *System* (3-S//H) *is ES if and only if A is* SS // HS, *i.e*.

$$\sigma(A) \subset \mathbf{C}_S \;//\; \mathbf{C}_H. \qquad \text{(17-S//H)}$$

Remark 7. Statements (i)–(iv) in Theorem 2 represent *sufficient conditions* for the ES of the linear system (3-S//H). Indeed:

(i) If (7-S//H) holds, then (14-S//H) is true for $\|\ \| = \|\ \|_p^D$ and $\delta(\alpha) = \alpha$. Indeed, for arbitrary $\alpha > 0$ in (14-S//H), by taking $0 < \varepsilon < \alpha$ in (7-S//H), we get $\|x_0\| < \alpha \Rightarrow \|x(t;t_0,x_0)\| < \alpha r^{(t-t_0)}$ // $\|x(t;t_0,x_0)\| < \alpha \mathrm{e}^{r(t-t_0)}$.

(ii) If (9-S) // (12) – equivalently (9-H) holds, then (15-S//H) is true for $\|\ \| = \|\ \|_p^D$, $\gamma = 1$ and $M(\beta) = 1$.

(iii) If (10-S//H) holds, then (16-S//H) is true for $\|\ \| = \|\ \|_p^D$ and $K = 1$.

(iv) If (11-S//H) holds, then (17-S//H) is true, according to Remark 2.

Consequently, DIES_p of the linear time-invariant system (3-S//H) is a special (refined) type of ES, which incorporates information about the existence of invariant sets.

Theorem 2 has the following direct consequence:

Corollary 2. *Let* $1 \leqslant p \leqslant \infty$. *System* (3-S//H) *is* DIES_p *if and only if the matrix A is* SDS_p // HDS_p.

Proof. It results from Definitions 1, 2 and Theorem 2(i) ⇔ (iv). □

Remark 8. The equivalence between DIES_p (as a DT // CT system property) and SDS_p // HDS_p (as a matrix property) enlarges the role of the algebraic instruments in the qualitative analysis of the dynamical systems. The idea of using matrix norms // measures in order to refine the exploration of linear system behavior has appeared in some previous works, among which [1,6] deserve special attention. Theorem 3 and its corollary in Chapter III of [1] provide inequalities of general interest, not strictly related to ES or invariant sets, but they can be used to prove the implications (H) (iv) ⇒ (ii) and (H) (iv) ⇒ (iii) of our Theorem 2. Paper [6] studies the connections between ES and invariant sets of DT // CT systems, showing that matrix norms // measures can be used to characterize the set invariance property. Unlike our approach, in [6] the invariant sets are considered constant, not depending on time. Therefore, the ES and the set invariance are regarded as two distinct properties, without any comment on their possible merging for defining a stronger type of ES.

Remark 9. The control literature contains papers discussing the "*componentwise exponential asymptotic stability* (CWEAS)", which has been defined as follows. System (3-S//H) is CWEAS if there exist $d_i^+ > 0$, $d_i^- > 0$, $i = 1,\dots,n$, $0 < r < 1$ // $r < 0$ such that $\forall t, t_0 \in \mathbf{Z}_+$ // $\mathbf{R}_+$, $t \geqslant t_0$, $-d_i^- \leqslant x_i(t_0) \leqslant d_i^+ \Rightarrow -d_i^- r^{(t-t_0)} \leqslant x_i(t) \leqslant d_i^+ r^{(t-t_0)}$ // $-d_i^- \mathrm{e}^{r(t-t_0)} \leqslant x_i(t) \leqslant d_i^+ \mathrm{e}^{r(t-t_0)}$, $i = 1,\dots,n$, where $x_i(t_0)$, $x_i(t)$ denote the components of the initial condition $x(t_0)$ of (3-S//H) and of the corresponding solution $x(t)$, respectively. In terms of the notations introduced by the current paper, the symmetrical CWEAS (meaning $d_i^- = d_i^+$ in the above definition) has been characterized by "A^H is HS" for CT linear systems [7–11], and by "A^S is SS" for DT linear systems [12]. In our recent work [11], for the symmetrical CWEAS of CT linear systems we have given three equivalent characterizations, similar to the statements (i), (ii), (iv) of the above Theorem 2 (part H) in the particular case $p = \infty$. Thus, paper [11] has opened the perspectives of a generalization for symmetrical CWEAS, developed by the current analysis of DIES$_p$ with $1 \leqslant p \leqslant \infty$.

Remark 10. A DT // CT system can be DIES$_p$ for different r's and D's. All these constants r play the role of decreasing rates for the time-dependent invariant sets $X_p^{\varepsilon}(t; t_0)$ defined by (8-S//H). Thus, according to Theorem 2(iv), we define the *fastest decreasing rate* as $r_p^*(A) = \inf_{D \succ 0 \text{ diagonal}} \|A\|_p^D$ // $r_p^*(A) = \inf_{D \succ 0 \text{ diagonal}} m_p^D(A)$. Moreover, the positive value $1 - r_p^*(A)$ (DT case) // $|r_p^*(A)|$ (CT case) can be regarded as the DIES$_p$ *degree of system* (3-S//H), by using the analogy with the ES *degree of system* (3-H//S) defined as $1 - \max_{i=1,\ldots,n} |\lambda_i(A)|$ (DT case) // $|\max_{i=1,\ldots,n} \text{Re}\{\lambda_i(A)\}|$ (CT case) (e.g. [13]). For a DT // CT system, $\forall p$, $1 \leqslant p \leqslant \infty$, the DIES$_p$ degree cannot exceed the ES degree, by Remark 2.

Corollary 3. *The following three statements are equivalent*:

(i) *System* (3-S//H) *is* DIES$_1$.
(ii) *System* (3-S//H) *is* DIES$_\infty$.
(iii) *System* (3-S//H) *is* DIES$_p$ *for all* $1 \leqslant p \leqslant \infty$.

Proof. It is a direct consequence of Theorem 1 and Corollary 2. □

Remark 11. For a DT // CT system which is DIES$_p$ for all $1 \leqslant p \leqslant \infty$, the fastest decreasing rate $r_p^*(A)$ introduced in Remark 10 can have different values for different p. If A is irreducible, then (S) $\max_{i=1,\ldots,n} |\lambda_i(A)| \leqslant r_p^*(A) \leqslant r_1^*(A) = r_\infty^*(A) = \lambda_{\max}(A^S)$; (H) $\max_{i=1,\ldots,n} \text{Re}\{\lambda_i(A)\} \leqslant r_p^*(A) \leqslant r_1^*(A) = r_\infty^*(A) = \lambda_{\max}(A^H)$. Indeed, $\max_{i=1,\ldots,n} |\lambda_i(A)| \leqslant r_p^*(A)$ // $\max_{i=1,\ldots,n} \text{Re}\{\lambda_i(A)\} \leqslant r_p^*(A)$ (by Remark 2), $r_p^*(A) \leqslant r_p^*(A^S)$ // $r_p^*(A) \leqslant r_p^*(A^H)$ (by Lemma 4), $r_p^*(A^S) = \lambda_{\max}(A^S)$ // $r_p^*(A^H) = \lambda_{\max}(A^H)$ (by Remark 3), and, for $p = 1, \infty$, $r_p^*(A) = r_p^*(A^S)$ // $r_p^*(A) = r_p^*(A^H)$ (by Remark 4). Note that, when A is irreducible, Remarks 3 and 4 provide a procedure for constructing the diagonal matrix $D \succ 0$ such that $r_p^*(A) = \lambda_{\max}(A^S)$, $p = 1, \infty$.

Corollary 4. *If A is nonnegative // essentially nonnegative*, *then the following three statements are equivalent*:

(i) *System* (3-S//H) *is ES*.
(ii) *There exists a* p, $1 \leqslant p \leqslant \infty$, *such that system* (3-S//H) *is* DIES$_p$.
(iii) *System* (3-S//H) *is* DIES$_p$ *for all* $1 \leqslant p \leqslant \infty$.

Proof. This is a direct consequence of Corollaries 1, 2 and Proposition 2. □

Remark 12. If the matrix A of the system (3-S//H) is nonnegative // essentially nonnegative and SS // HS, then $r_p^*(A) = \lambda_{\max}(A), \forall p, 1 \leqslant p \leqslant \infty$, as resulting from the proof of Lemma 3. When A is irreducible, Remark 3 ensures a construction procedure for the diagonal matrix $D \succ 0, \forall p$, $1 \leqslant p \leqslant \infty$.

Remark 13. Corollary 3 and Remark 12 provide a generalization of Theorem 7.4(iv) in [14], which proves two properties of CT systems that can be related to our DIES_1 and DIES_∞. For A essentially nonnegative and HS, work [14] uses the equivalent characterization "A is a $-M$ matrix", $\lambda_{\max}(A)$ is referred to as the "importance value of A", and the invariant sets are called "exponentially contractive" with coefficient $\lambda_{\max}(A)$. However, at the conceptual level, [14] does not merge the properties of ES and set invariance in the sense of our DIES_p. For A both irreducible and reducible, [14] constructs the invariant sets from the left and right eigenvectors of A associated with $\lambda_{\max}(A)$, which may contain 0 elements, when A is reducible. Thus, the similarity between the approach in [14] and our DIES_1, DIES_∞ appears only for A irreducible and it is limited to the invariant sets with the fastest decreasing rate, i.e. in our notations, $r_1^*(A) = r_\infty^*(A) = \lambda_{\max}(A)$.

The use of A^S // A^H for testing the SDS_p // HDS_p of the matrix A suggests considering the DT // CT dynamical system defined by

$$\begin{aligned} &y(t+1) = A^S y(t) \;//\; \dot{y}(t) = A^H y(t), \quad t, t_0 \in \mathbf{Z}_+ \;//\; \mathbf{R}_+, t \geqslant t_0, \\ &\text{with } y(t_0) = y_0 \in \mathbf{R}^n \text{ initial condition,} \end{aligned} \tag{18-S//H}$$

as a *comparison system* for studying the DIES_p of system (3-S//H).

Corollary 5. *Let* $0 < r < 1 \,//\, r < 0$ *and* $D \succ 0$ *diagonal*

(i) *Let* $p = 1, \infty$. *The comparison system* (18-S//H) *is* DIES_p *for r and D if and only if system* (3-H//S) *is* DIES_p *for r and* D.
(ii) *Let* $1 < p < \infty$. *If the comparison system* (18-S//H) *is* DIES_p *for* r *and* D, *then system* (3-S//H) *is* DIES_p *for r and* D.

Proof. This follows from Theorem 2 together with Remark 4 for (i) and Lemma 4 for (ii). □

5. Example

Consider the matrix A defined by

$$A = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}, \quad a > 0, \quad b > 0, \tag{19}$$

which is HS.

The matrix A is HDS_p, $p = 1, \infty$, if and only if $a > b$. Indeed, for $D = \mathrm{diag}\{d_1, d_2\}, d_1, d_2 > 0$, we have $m_p^D(A) = \max\{-a + bd_2/d_1, -a + bd_1/d_2\}, p = 1, \infty$, and $m_p^D(A) < 0$ if and only if $b/a < d_1/d_2 < a/b$. The same condition results from Theorem 1, since

$$A^H = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix}, \tag{20}$$

is HS if and only if $a > b$.

The condition $a > b$ is sufficient for A to be HDS_p, $1 \leqslant p \leqslant \infty$ (according to Theorem 1), but it may not be necessary, as shown below for $p = 2$.

The matrix A is HDS_2 for any $a > 0, b > 0$. Indeed, for $D = \mathrm{diag}\{d_1, d_2\}, d_1, d_2 > 0$, we have $m_2^D(A) = \max\{-a + b(d_2/d_1 - d_1/d_2)/2, -a - b(d_2/d_1 - d_1/d_2)/2\} < 0$, which is equivalent to $\sqrt{(a/b)^2 + 1} - (a/b) < d_1/d_2 < \sqrt{(a/b)^2 + 1} + (a/b)$.

Consider the CT linear system (3-H) with A defined by (19). System (3-H) is ES for any $a > 0$, $b > 0$, and the ES degree is a. If $a > b$, system (3-H) is DIES_p, $1 \leqslant p \leqslant \infty$, by Corollary 2. The condition $a > b$ is also necessary for DIES_p, $p = 1, \infty$. System (3-H) is DIES_2 for any $a > 0$, $b > 0$.

If $p = 1, \infty$, then $r_p^*(A) = -a + b = \lambda_{\max}(A)$ corresponds to $d_1 = d_2$, which means the invariant sets are squares, as a particular type of diamonds ($p = 1$) or rectangles ($p = \infty$). The DIES_p degree is $a - b$.

If $p = 2$, then $r_2^*(A) = -a < \lambda_{\max}(A)$ corresponds to $d_1 = d_2$, which means the invariant sets are circles, as a particular type of ellipses. The DIES_2 degree is a.

Acknowledgments

The authors are grateful to the referees for their comments and suggestions.

References

[1] W.A. Coppel, Stability and Asymptotic Behavior of Differential Equations, D.C. Heath and Company, Boston, 1965.
[2] E. Kaszkurewicz, A. Bhaya, Matrix Diagonal Stability in Systems and Computation, Birkhäuser, Boston, 2000.
[3] A. Michel, K. Wang, Qualitative Theory of Dynamical Systems, Marcel Dekker, Inc., New York–Bassel–Hong Kong, 1995.
[4] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[5] J. Stoer, C. Witzgall, Transformations by diagonal matrices in a normed space, Numer. Math. 4 (1962) 158–171.
[6] H. Kiendl, J. Adamy, P. Stelzner, Vector norms as Lyapunov functions for linear systems, IEEE Trans. Automat. Control 37 (1992) 839–842.
[7] M. Voicu, Componentwise asymptotic stability of linear constant dynamical systems, IEEE Trans. Automat. Control 29 (1984) 937–939.
[8] M. Voicu, Free response characterization via flow-invariance, Preprints of the 9th World Congress of IFAC, Budapest, vol. 5, 1984, pp. 12–17.
[9] M. Voicu, On the application of the flow-invariance method in control theory and design, Preprints of the 10th World Congress of IFAC, München, vol. 8, 1987, pp. 364–369.
[10] A. Hmamed, A. Benzaouia, Componentwise stability of linear systems: a non-symmetrical case, Int. J. Robust Nonlinear Control 7 (1997) 1023–1028.
[11] O. Pastravanu, M. Voicu, On the componentwise stability of linear systems, Int. J. Robust Nonlinear Control 15 (2005) 15–23.
[12] A. Hmamed, Componentwise stability of 1-D and 2-D linear discrete systems, Automatica 33 (1997) 1759–1762.
[13] V. Balakrishnan, S. Boyd, S. Balemi, Computing the minimum stability degree of parameter-dependent linear systems, in: S.P. Bhattacharyya, L.H. Keel (Eds.), Control of Uncertain Dynamic Systems, CRC Press, 1991, pp. 359–378.
[14] L. Gruyitch, J.-P. Richard, P. Borne, J.-C. Gentina, Stability Domains, Chapman & Hall/CRC, Boca Raton, 2004.